JOURNAL OF ALGEBRA 61, 16–27 (1979)

Subgroups of Finite Groups of Lie Type

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Communicated by Walter Feit

Received January 9, 1979

1. INTRODUCTION

Let $G = G(q)$ be a finite group of Lie type defined over the field F_q. Choose a Borel subgroup, $B = UH \leqslant G$, of G, where U is unipotent and H a Cartan subgroup of G. In this paper we are concerned with the subgroups, Y, of G, such that $H \leqslant Y$, and we determine these subgroups in the case where q is odd and $q > 11$.

Associated with G is a root system, Σ, and a collection of root subgroups $\{U_\alpha : \alpha \in \Sigma\}$ such that $U = \prod_{\alpha \in \Sigma^+} U_\alpha$ and such that $H \leqslant N(U_\alpha)$ for each $\alpha \in \Sigma$. In Lemma 3 of [11] it was shown that for $q > 4$ any H-invariant subgroup of U is essentially a product of root subgroups (the word "essentially" is relevant only when G is twisted, with some root subgroup non-Abelian). This result was extended in [7], where it was shown that any unipotent subgroup of G normalized by H is of this form (although now negative roots are allowed).

THEOREM. *Suppose q is odd and $q > 11$. Let $H \leqslant Y \leqslant G$ and set $Y_0 = \langle U_\alpha \cap Y \mid \alpha \in \Sigma \rangle$. Then*

(i) $Y_0 \trianglelefteq Y$ *and* $Y = Y_0 N_Y(H)$;

(ii) $Y_0 = U_0 X_0$, *where* $U_0 \cap X_0 = 1$, U_0 *is unipotent and* X_0 *is a central product of groups of Lie type*;

(iii) U_0 *and each component of* X_0 *is generated by groups of the form* $U_\alpha \cap Y$, $\alpha \in \Sigma$; *and*

(iv) *If* $G \neq {}^2G_2(q)$, *then for* $\alpha \in \Sigma$, $U_\alpha \cap Y = 1$, U_α, *or* $\Phi(U_\alpha)$.

We refer the reader to (2.6) for the explicit description of the group Y_0. The above theorem can be viewed as the completion (at least for fields of odd characteristic) of the work in [11, 7], or as the first step in determining those subgroups of G that contain a maximal torus of G. From the latter point of view, the proof

* Research supported in part by NSF Grants MCS 76-07015 and MCS 78-01944.

0021-8693/79/110016-12$02.00/0

of the theorem is somewhat unsatisfactory as it involves the use of certain classification theorems for simple groups and arguments outside the scope of the Lie theory. It would be preferable to have a proof that only used the Lie theory and made effective use of analogous results for algebraic groups.

One pleasant aspect of the result is the way Y_0 is described. The point here is that the group Y_0 is defined in terms of a fixed root system, so that all questions concerning the structure of Y_0 can be answered simply by looking at this root system and appropriate commutator relations.

For $q \leqslant 11$ there are infinitely many exceptions to the theorem, although one could probably determine these with a bit of extra work. When q is a (suitably large) power of 2, one should be able to prove the same result. Many of our arguments are independent of the characteristic of the underlying field, but in a couple of places we used results on 2-fusion which at present do not exist for odd primes, and these would be required in order to generalize the proof presented here. A final remark would be that when dealing only with classical groups one could probably get a similar result using properties of the underlying module. The groups involved could be described in terms of their action on that module.

We fix the group $G = G(q)$, H a Cartan subgroup of G, and $B = UH$ a Borel subgroup of G. Without loss of generality, we may assume that G is a universal group. Let bars denote images in $\bar{G} = G/Z(G)$. As before, we let Σ be the root system of G, and for $H \leqslant Y \leqslant G$ set $Y_0 = \langle U_\alpha \cap Y_0 \mid \alpha \in \Sigma \rangle$. In case U_α is not elementary Abelian, we let $V_\alpha = \Phi(U_\alpha)$. Let $q = p^a$ for $p > 2$, a prime. Finally, we let $W = N/H$ be the Weyl group of G.

In the case of $G \simeq SU(n, q)$ with $n = 2l + 1$ the system Σ is actually the union of B_l and C_l. We interpret this for root groups as follows. Let $\alpha \in \Sigma$ be such that U_α is non-Abelian. Then regard α as a short root in B_l and let $U_{2\alpha} = \Phi(U_\alpha)$. So $U_{2\alpha}$ is a long root in C_l. With this interpretation, we have a root subgroup for each root in Σ.

2. Preliminaries

(2.1) *Let $B \leqslant P$ (so that P is a parabolic subgroup of G) and assume that $H \leqslant P^g$ for $g \in G$. Then H is a Cartan subgroup of P^g. That is, H is conjugate in P^g to a subgroup of B^g.*

Proof. Let $g = l_1 w l_2$, where $l_1, l_2 \in B$ and $w \in N$. Then $H \leqslant P^{wl_2}$, so $H^{l_2^{-1}} \leqslant P^w$. Also, $H \leqslant P^w$, so $\langle H, H^{l_2^{-1}} \rangle \leqslant P^w \cap B$. Thus H is an Abelian Hall subgroup of $\langle H, H^{l_2^{-1}} \rangle$, so $H^{l_2^{-1}c} = H$ for some $c \in \langle H, H^{l_2^{-1}} \rangle$. Now, H is a Cartan subgroup of P^w and $c \in P^w$, so $H^{l_2^{-1}}$ is a Cartan subgroup in P^w. The lemma follows.

(2.2) *Suppose* $q > 4$ *and* $U_0 \leqslant U$ *is H-invariant. Then* $U_0 = \prod_{\alpha\in\Sigma^+}(U_\alpha \cap U_0)$ *and if* $G \neq {}^2G_2(q)$, *then* $U_\alpha \cap U_0 = 1$, V_α, *or* U_α, *for each* $\alpha \in \Sigma^+$.

Proof. This is Lemma 3 of [11].

(2.3) *Let* $X \leqslant H$, $X \trianglelefteq N$, *such that* $C_U(X) = 1$. *Then* $N_G(X) = N$. *In particular, if* $q \geqslant 4$, *then* $N_G(H) = N$.

Proof. The second statement is immediate from the first. Just consider the action of H on the root subgroups in U. For the first statement assume that $C_U(X) = 1$ and $g \in N_G(X)$. Write $g = u_1wu_2$ with $u_1, u_2 \in U$ and $w \in N$. We do this so that $u_2 \in U_w^- = U \cap U^{w_0w}$, where w_0 is in the coset of the long word in the fundamental reflections that generate W. For $x \in X$, $g^x = g[g, x] \in gX$. On the other hand, $g^x = u_1^xw^xu_2^x = u_1^xw[w, x]\, u_2^x$. By the uniqueness of the Bruhat decomposition we have $u_1 = u_1^x$ and $[w, x]\, u_2^x = u_2[g, x]$. The latter equation leads to $(x^{-1})^w\, u_2 = u_2(x^{-1})^g$. Now $(x^{-1})^w$, $(x^{-1})^g$ are both in X and, modulo U, they are equal. So $(x^{-1})^w = (x^{-1})^g$ and $(x^{-1})^g \in C(u_2)$. We conclude that $X \leqslant C(u_1) \cap C(u_2)$, so $g = w \in N$, proving the result.

(2.4) (Cline–Parshall–Scott.) *Suppose* $q > 4$ *and* V *is an H-invariant p-subgroup of G. Then* $V = \prod_{\alpha\in\Sigma}(U_\alpha \cap V)$, *and if* $G \neq {}^2G_2(q)$, *then* $U_\alpha \cap V = 1$, V_α, *or* U_α, *for each* $\alpha \in \Sigma$.

Proof. By (3.12) of [6] there is a parabolic subgroup $P \geqslant B$ and an element $g \in G$ such that $V \leqslant O_p(P^g)$ and $H \leqslant P^g$. By (2.1) we may assume that $H \leqslant B^g$. So $H^{g^{-1}} \leqslant B$ and $H^{g^{-1}} = H^u$ for some $u \in U$. Then $ug \in N_G(H) = N$ (by (2.3)), so $B^g = B^{ug} = B^w$, for some $w \in N$. The result now follows from (2.2) and the fact that N permutes $\{U_\alpha \mid \alpha \in \Sigma\}$.

(2.5) *Suppose* $q - 1$ *is not a power of* 2 *and let* $n \in N - H$ *be a 2-element. Then* $[n, K] \neq 1$, *where K is the product of the Sylow r-subgroups of H, for odd primes* $r \mid q - 1$.

Proof. Suppose $[n, K] = 1$ with n a 2-element. We may assume that G is an untwisted group. For otherwise we can look at a subgroup (in fact the fixed points of an automorphism of G) having the same Weyl group and containing K. Also, we may replace G by $\bar{G}$. Finally, we replace $\bar{G}$ by $\hat{G}$, where $\hat{G}$ is $\bar{G}$ together with all diagonal automorphisms of $\bar{G}$. Then $\bar{K}$ is contained in $\hat{K}$, a Hall subgroup of a Cartan subgroup, $\hat{H} \geqslant \bar{H}$, and we have $[\bar{n}, \hat{K}] = 1$ (as $\bar{n}$ is a 2-element centralizing $\hat{K}/\bar{K}$ and $\bar{K}$). At this point an easy check of the action of $\bar{n}$ on $\hat{K}$ gives a contradiction. Just choose a fundamental root α_i, such that $\alpha_i^{\bar{n}} \neq \alpha_i$, and construct an element of $\hat{K}$ not centralized by $\bar{n}$.

(2.6) *Let* $H \leqslant Y \leqslant G$ *and assume* $q > 4$. *Let* $\Delta = \{\alpha \in \Sigma \mid U_\alpha \cap Y \neq 1\}$ *and* $\sim$ *the smallest equivalence relation on* Δ *such that* $\alpha \sim \beta$ *if* $\langle U_\alpha \cap Y, U_{-\alpha} \cap Y\rangle$

and $\langle U_\beta \cap Y, U_{-\beta} \cap Y\rangle$ *do not commute. Let* $\Delta_1, \ldots, \Delta_l$ *be the equivalence classes of* Δ *under* $\sim$. *For* $i = 1, \ldots, l$, *let* $\bar{\Delta}_i = \{\alpha \in \Delta_i \mid -\alpha \in \Delta_i\}$, $Y(\Delta_i) = \langle U_\alpha \cap Y \mid \alpha \in \Delta_i\rangle$, *and* $Y(\bar{\Delta}_i) = \langle U_\alpha \cap Y \mid \alpha \in \bar{\Delta}_i\rangle$.

(a) *For* $i = 1, \ldots, l$, $Y(\bar{\Delta}_i)$ *has a* (B, N)*-pair and* $\prod_{\alpha \in \Sigma^+ \cap \bar{\Delta}_i} (U_\alpha \cap Y) = U \cap Y(\bar{\Delta}_i)$ *is a Sylow* p*-subgroup of* $Y(\bar{\Delta}_i)$.

(b) *For* $i = 1, \ldots, l$, *let* $\Delta_i^0 = \Delta_i - \bar{\Delta}_i$. *Then* $Y(\bar{\Delta}_i)$ *normalizes* $U_i = \prod_{\alpha \in \Delta_i^0} (U_\alpha \cap Y)$, U_i *is a* p*-group, and* $Y(\Delta_i) = Y(\bar{\Delta}_i)\, U_i$.

(c) Y_0 *is a commuting product of the groups*, $Y(\Delta_i)$.

(d) $Y_0/O_p(Y_0)$ *is a commuting product of groups of Lie type.*

Proof. A direct check gives the result when $G = {}^2G_2(q)$. From now on we assume that $G \neq {}^2G_2(q)$. If $G \simeq SU(n, q)$ for n odd, then $\Sigma = B_l \cup C_l$, for $l = \frac{1}{2}(n-1)$.

We recall that there is a root subgroup for each root in Σ. If $\alpha \in \Sigma$, then by (2.4) $U_\alpha \cap Y = 1$ or U_γ for some $\gamma \in \Sigma$.

If $\alpha \in \bar{\Delta}_i$, then $Y(\bar{\Delta}_i) \geqslant \langle U_{\pm\alpha}\rangle$ and hence $Y(\bar{\Delta}_i)$ contains a coset representation of the reflection $s_\alpha \in W$. It follows that $\bar{\Delta}_i$ is a root system. Let $N(\bar{\Delta}_i) = \langle N_\alpha \mid \alpha \in \bar{\Delta}_i\rangle$, where $N_\alpha = N \cap \langle U_{\pm\alpha}\rangle$. Let $U(\bar{\Delta}_i) = \pi U_\alpha$, where the product ranges over $\bar{\Delta}_i \cap \Sigma^+$. Then $U(\bar{\Delta}_i)$ is a group and it is straightforward to check that $X = U(\bar{\Delta}_i)\, N(\bar{\Delta}_i)\, U(\bar{\Delta}_i)$ is a group with a (B, N)-pair. (To see this, first choose a fundamental system of roots in $\bar{\Delta}_i$ and then consider the corresponding fundamental reflections in $N(\bar{\Delta}_i)/(N(\bar{\Delta}_i) \cap H)$.) This gives (a).

Now consider U_i. Suppose $\alpha \in \Delta_i^0$, $\beta \in \Delta$ and $[U_\alpha \cap Y, U_\beta \cap Y] \neq 1$. We may assume $U_\alpha \cap Y = U_\alpha$ and $U_\beta \cap Y = U_\beta$. The commutator relations imply that, with the possible exception of $G = G_2(3^a)$, we have $[U_\alpha, U_\beta] \geqslant U_{\alpha+\beta}$, hence $\alpha + \beta \in \Delta$. Therefore, β and $\alpha + \beta$ are in Δ_i. If $\alpha + \beta \in \bar{\Delta}_i$, then (excluding the case $G = G_2(3^a)$) $[U_{-(\alpha+\beta)}, U_\beta] \geqslant U_{-\alpha}$, a contradiction. Thus $\alpha + \beta \in \Delta_i^0$, and this implies (b) and (c). The same argument works for $G = G_2(3^a)$ unless $[U_{-(\alpha+\beta)}, U_\beta] = 1$, and here a direct check yields the result.

To prove (d) we need only identify each $Y(\bar{\Delta}_i)$ as a group of Lie type. If $Y(\bar{\Delta}_i) = \langle U_{\pm\alpha}\rangle$ for some α, then this is immediate. Otherwise, use Theorem C of [9].

(2.7) *If* $G = {}^2G_2(q)$, *then the theorem holds.*

Proof. This can be done by a direct check or by using some of the properties in Section 2 of [15].

(2.8) (a) $C_N(O_2(H))/H$ *is a* 2*-group.*

(b) *If* $H \leqslant X \leqslant N$, *then* H *is characteristic in* X.

Proof. Choose a maximal set of long roots in Σ^+, say $\gamma_1, \ldots, \gamma_k$, with the property that for each $i \neq j$ in $\{1, \ldots, k\}$ we have $[J_i, J_j] = 1$, where $J_i =$

$\langle U_{\pm\gamma_i}\rangle \simeq SL(2, q)$, for $i = 1,..., k$. Let $\langle t_i\rangle = Z(J_i)$. Then $C_G(\langle t_1 ,..., t_k\rangle) = J_1 \cdots J_kHD$, where $D = 1$ or $D = \langle U_{\pm\beta}\rangle$, for some short root $\beta \in \Sigma$ (see (4.2) of [16]). Since$C_N(O_2(H)) \leqslant C_G(\langle t_1 ,..., t_k\rangle)$, we easily have (a).

For (b), let $\sigma \in \mathrm{Aut}(X)$, where $H \leqslant X \leqslant N$. First suppose that $q - 1$ is not a power of 2. We have $H^\sigma \trianglelefteq X$, so $[O_2(H^\sigma), O(H)] = 1$. By (2.5) we conclude that $O_2(H^\sigma) = O_2(H)$. Now suppose that $q - 1 = 2^a$. As $q > 11$ we have $q - 1 \geqslant 2^4$. The group HH^σ is nilpotent of class at most 2, so $(h^\sigma)^2 \in C(\Omega_1(O_2(H)))$ for each $h \in H$. So $(H^2)^\sigma \leqslant C_G(\langle t_1 ,..., t_k\rangle)$, and as $H(H^2)^\sigma$ has class at most 2, we see that $(H^2)^\sigma \leqslant H$. So $\Omega_1(O_2(H)^\sigma) = \Omega_1(O_2(H^2)^\sigma) = \Omega_1(O_2(H))$. So in either case, $H^\sigma \leqslant C(\Omega_1(O_2(H))) \leqslant C_G , \langle t_1 ,..., t_k\rangle$, and we check that $H^\sigma = H$.

3. Proof of the Theorem

In this section we prove the theorem. Suppose the result false, let $H \leqslant Y < G$, with Y a counterexample and $|Y| \cdot |G|$ minimal. By (2.6) conclusion (i) of the theorem is false. By (2.7) we may assume that $G \neq {}^2G_2(q)$. Set $Y_0 = \langle U_\alpha \mid U_\alpha \leqslant Y\rangle$. Recalling the convention regarding root subgroups of odd-dimensional unitary groups, and applying (2.2), we have $Y_0 = \langle И_Y(H, p)\rangle$.

(3.1) *Let $x \in H - Z(G)$. Then $C_Y(x) = C_{Y_0}(x)\, N_{C_Y(x)}(H) \leqslant N_Y(Y_0)$.*

Proof. It suffices to obtain the factorization of $C_Y(x)$, as the inequality is then obvious. If $C_Y(x) < Y$, then we are done by minimality of Y. So we assume that $x \in Z(Y)$. The group $C_G(x)$ has the following properties: $C_G(x) \trianglerighteq D_1 \cdots D_hT$, a central product, where each D_i is a group of Lie type defined over an extension field of F_q, and $T \leqslant H$. One reference for this is (2.5) of [12], where it is also noted that each of the factors is normal in $C_G(x)$. By (2.12) of [12], $C_G(x) = D_1 \cdots D_hH$.

For $i = 1,..., h$, let $H_i = H \cap D_i$. Then H_i is a Cartan subgroup of D_i (see the proof of (2.6) of [12], then pass to the overlying algebraic group and use (4.1) of [13]). Also, $Y = H\bar{Y}$ for $\bar{Y} = Y \cap D_1 \cdots D_h$, and for $i = 1,..., h$ we let Y_i be the projection of $\bar{Y}$ to D_i. By minimality $Y_{i0} \trianglelefteq Y_i$ and $Y_i = Y_{i0}N_{Y_i}(H_i)$, for $i = 1,..., h$. Suppose that $U_\alpha \leqslant Y$. For each i, H normalizes the projection of U_α to D_i. It follows from (2.4) that $U_\alpha \leqslant D_i$ for some i. Therefore, $Y_0 = \prod_{i=1}^h Y_{i0}$.

Let $N_0 = \prod_{i=1}^h N_{D_i}(H_i)$. Then N_0 normalizes $C(H_1 \cdots H_h) \cap C_G(x) = H$. Consequently, $Y = \bar{Y}H \leqslant Y_0N_0H \leqslant Y_0N(H)$ and the result holds.

(3.2) (i) *If $R \in И(H)$ and $RH < Y$, then $HR = R_0N_{RH}(H) \leqslant Y_0N_Y(H)$.*

(ii) *Suppose $И_Y(H, p) \neq 1$. If $R \trianglelefteq Y$ with R a central product of Chevalley groups over extension fields of F_q, then $RH = R_0N_{RH}(H) \leqslant Y_0N_Y(H)$.*

(iii) *If* $H \leqslant R < Y$, *then* $R \leqslant Y_0 N_Y(H) \leqslant N_Y(Y_0)$.

(iv) *If* $x \in H - Z(G)$, *then* $C_Y(x) \leqslant N_Y(Y_0)$.

(v) *If* $y \in Y$, t, $t^y \in H$ *and* $t \neq t^y$, *then* $y \in Y_0 N_Y(H)$.

Proof. (i) follows by minimality of Y. For (ii) first note that the only difficulty is when $RH = Y$ (otherwise apply (i)), so each element of $\mathcal{U}_Y(H, p)$ is in R. Let $V \in \mathcal{U}_Y^*(H, p)$. By the Borel–Tits theorem ((3.12) of [6]), $V = O_p(P)$, where P is a parabolic subgroup of R. By (2.6) and (i) we know the structure of PH. We must have $V = (PH)_0$, so $VH \trianglelefteq PH$ and it follows that P is a Borel subgroup of R. Let J be an H-invariant Cartan subgroup of P. Then $H \leqslant N_Y(J) < Y$. Now, $R = \langle P, N_F(J)\rangle$, so (ii) follows from the above and (i).

Also, (iii) follows from (i), and (iv) follows from (3.1). To prove (v) suppose that t, $t^y \in H$ with $t \neq t^y$. Then $H, H^y \leqslant C_Y(t^y) \leqslant Y_0 N_Y(H) \cap (Y_0 N_Y(H))^y$, and we may assume that $Y_0 N_Y(H) < Y$. As $H \leqslant (Y_0 N_Y(H))^y$, (iii) implies that $(Y_0 N_Y(H))^y \leqslant Y_0 N_Y(H)$, so by orders we have equality. The conclusion follows from (2.8)(b) and (ii) applied to $Y_0 N_Y(H)\langle Y\rangle$.

(3.3) (i) $O_p(Y) = 1$.

(ii) *Suppose* $\mathcal{U}_Y(H, p) \neq \{1\}$. *Then Y contains no normal subgroup, D, with D a central product of Chevalley groups, each defined over an extension field of* F_q.

Proof. (i) Suppose $O_p(Y) = V > 1$. By (3.12) of [6], there is a parabolic subgroup $P < G$ such that $Y \leqslant P$ and $V \leqslant O_p(P)$. Now, (2.1) implies that H is a Cartan subgroup of P; hence of $P/O_p(P)$. Minimality, (2.4), and arguments similar to those in the proof of (3.1), forces $YO_p(P)/O_p(P) \leqslant N_P(H)Y_0\, O_p(P)/O_p(P)$. So $Y \leqslant O_p(P)\, N_P(H)Y_0$ and it follows that $Y = Y_0 N_Y(H)$, which we are assuming false.

(ii) Suppose Y contains such a subgroup $D \trianglelefteq Y$. By (3.2)(ii) $DH = X$ satisfies $X = X_0 N_X(H)$. By (2.6) H normalizes a Sylow p-subgroup, V, of X_0. Since $D = E(X) = X_0$, we have $Y = X_0 N_Y(V) \leqslant Y_0 N_Y(V)$. Since $N_Y(V) < Y$, minimality gives $Y = Y_0 N_Y(H)$, a contradiction.

(3.4) (i) *Y contains no root subgroup U_α for α a long root.*

(ii) $Y_0 = O_p(Y_0)$.

Proof. (i) Suppose $U_\alpha \leqslant Y$, with α a long root. By (3.3) and Baer's theorem (see [1]) there is an element $y \in Y$ such that $\langle U_\alpha, U_\alpha{}^y\rangle$ is not a p-group. Then $\langle U_\alpha, U_\alpha{}^y\rangle \cong SL(2, q)$, and setting $\langle t\rangle = Z(\langle U_\alpha, U_\alpha{}^y\rangle)$, t is a classical involution of G (see [2]). Let $X = \langle U_\alpha{}^Y\rangle$ and apply Corollary III of [2] to obtain the structure of $X/O(X)$. We know that for each $g \in G$ $\langle U_\alpha, U_\alpha{}^g\rangle$ is either a p-group or isomorphic to $SL(2, q)$. It follows from (3.3)(i) that $U_\alpha \leqslant C(O(X))$ and hence $O(X) \leqslant Z(X)$. At this point we have a contradiction to (3.3)(ii).

(ii) Suppose $Y_0 \neq O_p(Y_0)$. As $\mathcal{U}_Y(H, p) \neq 1$ we use (2.6) to see that Y_0 is generated by root subgroups, U_β, for $\beta \in \Sigma$ a short root. Suppose β is such a root and $U_{-\beta} \leqslant Y$. Note that $\langle U_{\pm\beta}\rangle \not\cong SU(3, q)$, for otherwise $\Phi(U_\beta) \leqslant Y$ and contradicts (i). Also, $\langle U_{\pm\beta}\rangle \cong SL(2, q^i)$, for some i, as G is a universal group. By (2.6), (i), and the commutator relations we have $\langle U_{\pm\beta}\rangle \trianglelefteq C_Y(t)$, where $\langle t\rangle = Z(\langle U_{\pm\beta}\rangle)$. However, $C_Y(t) \leqslant N(Y_0)$ (by (3.1)) and this implies that t is a classical involution in Y. So we can use the above argument once we show that $O(X) \leqslant Z(X)$, where $X = \langle U_\beta{}^Y\rangle$. Consider $O(X)H$, apply minimality, (2.6), and (3.3)(i). We conclude that $O(X) \leqslant N(H)$. So $[O(X), t] \leqslant O(X) \cap H$ and $[O(X), t, t] = 1$. But then $1 = [O(X), t]$ and since t is a classical involution of Y, $[O(X), \langle U_{\pm\beta}\rangle] = 1$. We then have $O(X) \leqslant Z(X)$, as required. We have now shown that for each β with $U_\beta \leqslant Y$, $U_{-\beta} \nleqslant Y$. From (2.6) we conclude that $Y_0 = O_p(Y_0)$.

By (3.4) and (3.3) $Y_0 = 1$ or $Y_0 \ntrianglelefteq Y$. Choose $g \in Y - Y_0N_Y(H)$,

$$T \in \mathrm{Syl}_2(H^g \cap N_Y(H)),$$

with $m(\bar{T})$ maximal.

(3.5) (a) *G has Lie rank at least* 2.

(b) $Y \neq O(Y)H$.

(c) $T \nleqslant Z(G)$.

Proof. (a) Recall that $G \neq {}^2G_2(q)$. It remains to consider $G \cong SL(2, q)$ and $SU(3, q)$, and here we verify the result by appealing to pp. 285–286 of [8] or the theorem in [5], respectively.

(b) Suppose $Y = O(Y)H$. If $m(O_2(\bar{H})) \geqslant 2$, then we can use (3.1) to obtain a contradiction. So suppose $m(O_2(\bar{H})) = 1$. This implies that G has Lie rank 2, $q \equiv -1 \pmod 4$, and $G = \mathrm{Sp}(4, q)$. Choose $M \trianglelefteq Y$ such that $M \leqslant O(Y)$ and M is minimal such that $Y = MH$. Let M/L be a chief factor of Y and let $t \in \mathrm{Inv}(H) - Z(G)$.

By (3.2)(iii), $LH \leqslant Y_0N_Y(H)$ and $O_p(L) \leqslant O_p(Y) = 1$ by (3.3). Consequently, $L \leqslant N_Y(H)$. But $N_G(H)/H$ is a 2-group, so $L \leqslant H$. Since $C_Y(L) \geqslant \langle L, H\rangle = Y$, we have $L \leqslant Z(Y)$. But (3.1) then implies that $L = 1$. Now, H acts irreducibly on M, so $O(H) \cap C(M) \neq 1$, and this again contradicts (3.1).

(c) Suppose $T \leqslant Z(G)$. We claim that $O_2(H)$ is strongly closed in a Sylow 2-subgroup of Y. Otherwise, there is a 2-element $h \in H$ and an element $y \in Y$ such that $h^y \in N(O_2(H)) - O_2(H)$. Using (a) we have $O_2(H) > O_2(H) \cap Z(G)$, and both h and h^y are in $N_Y(O_2(H))$. Here $O_2(H) \ntrianglelefteq Y$, and (3.2)(iii) implies that $h^y \in Y_0N_Y(H) = O_p(Y_0)\, N_Y(H)$, contradicting $T \leqslant Z(G)$. This proves the claim.

By the main theorem of [10] we know the structure of $X/O(X)$, where

$X = \langle O_2(H)^Y \rangle$. By the Frattini argument, $Y = XN_Y(O_2(H))$, so (3.2) implies that $Y = XH$ or $Y = N_Y(O_2(H))$. Suppose the latter holds. By (3.2)(v) and (3.1), $Y = Y_0N_Y(H)$, a contradiction. Therefore, $Y = XH$. Now $X/O(X) = (X_1/O(X)) \cdots (X_2/O(X))$, where each $X_i/O(X)$ is a 2-group or a covering group of $L_2(2^i)$, $U_3(2^i)$, $Sz(2^i)$, J_1, or $L_2(q_0)$, for $q_0 \equiv 3, 5 \pmod 8$. Since $H \leqslant C(O_2(H))$, H stabilizes each X_i, and minimality implies $Y = X_iH$, for some i.

By (b), $O(X)H < Y$, so (3.2) implies that $O(X)H \leqslant Y_0N_Y(H)$. Suppose $Y_0 = 1$. Then $O(X) \leqslant N(H)$, so $[O_2(H), O(X)] \leqslant O_2(H) \cap O(X) = 1$. Thus $O_2(H) \leqslant C(O(X))$ and hence $O(X) \leqslant Z(X)$. As above (3.1) and (3.2) imply that X_i is quasi-simple. But $\bar{H}$ acts faithfully on $\bar{X}_i$ (otherwise, $H \cap Z(Y) \neq Z(G)$, contradicting (3.1)), inducing an Abelian group that centralizes $O_2(H) \cap X_i$. Since $q > 11$ we must have $O_2(\bar{H}) = \Omega_1(O_2(\bar{H}))$, $X/O(X) \cong U_3(2^i)$, and $O(\bar{H})$ cyclic of order dividing $2^i + 1$. But $O(\bar{H})$ is not cyclic (by (a)).

Now assume $Y_0 \neq 1$. As $O(X)H \leqslant Y_0N_Y(H)$ and $O_p(Y) = 1$, we have $O(X) \cap Y_0 = 1$. So $Y_0 \cong Y_0O(X)/O(X)$ is a p-group normalized by $O_2(H)\, O(X)/O(X)$. The only possibility is $X/O(X) \cong U_3(2^i)$ with $|O_2(H)| = 2^i$ and Y_0 cyclic of order dividing $2^i + 1$. But H acts without fixed points on Y_0 and centralizes $O_2(H)$. As in the previous paragraph HY_0 is cyclic of order divisible by $2^i + 1$. This is a contradiction.

(3.6) *$\bar{Y}_0\bar{H}$ is r-tightly embedded in $\bar{Y}$, for all primes r dividing $|\bar{H}|$.*

Proof. Suppose $g \in \bar{Y} - N(\bar{Y}_0\bar{H})$ with $|\bar{Y}_0\bar{H} \cap (\bar{Y}_0\bar{H})^{\bar{g}}|$ divisible by r. Then there exists an element, h, of order r with $h \in H - Z(G)$ and an element $u \in Y_0$ such that $\bar{h}^{\bar{g}\bar{u}} \in \bar{H}$. By (3.2)(v), $h^{gu} = h^j$ for some $j \in Y_0N_Y(H)$. So $guj^{-1} \in C_Y(h) \leqslant Y_0N_Y(H)$ and $g \in Y_0N_Y(H) \leqslant N(Y_0H)$, a contradiction.

(3.7) *If $m(\bar{T}) > 1$, then $O_r(H) = 1$ for each odd prime r dividing $q - 1$.*

Proof. Suppose $m(\bar{T}) > 1$. Then $O(H) = \langle C(t) \cap O(H) \mid t \in T - Z(G) \rangle$, so $O(H) \leqslant Y_0{}^gN_Y(H^g)$, by (3.1). Suppose $O_r(\bar{H})$ is not cyclic. Then (3.2)(v) and (3.1) imply that $Y_0{}^gO_2(H^g) \leqslant \langle C(t) \cap Y_0{}^gO_2(H^g) \mid t \in O(H) - Z(G) \rangle \leqslant Y_0N_Y(H)$. So we may assume $T \in \mathrm{Syl}_2(H^g)$ and $[T, O(H)] \leqslant O(H) \cap Y_0{}^gO_2(H^g) = 1$ (by orders). Then $T \leqslant C(O(H))$ so (2.5) implies that $O_r(H) = 1$ for each odd prime r dividing $q - 1$. Now suppose $O(\bar{H})$ is cyclic. G has Lie rank at least 2, and we may assume that $q - 1$ is not a power of 2. It follows that $G \cong SL(3, q)$ and $q - 1 = 3 \cdot 2^a$ for some a. However, this contradicts (3.6) and the assumption $m(\bar{T}) > 1$.

(3.8) $m(\bar{T}) > 1$.

Proof. By (3.5)(c), $m(\bar{T}) \geqslant 1$. Suppose $m(\bar{T}) = 1$ and let $\Omega_1(\bar{T}) = \langle \bar{t} \rangle$. If $\bar{h}$ is an involution in $\bar{H}$ with $\bar{h}^{\bar{t}} = \bar{h}$, then $h^t = hz$ for $z \in Z(G)$ and $(t^{-1})^h =$

$t^{-1}z \in T \leqslant H^g$. By (3.6), $\bar{h}$ normalizes $(\bar{Y}_0\bar{H})^{\bar{g}}$. So by choice of $\bar{T}$, we have $C(\bar{t}) \cap O_2(\bar{H})$ cyclic. Consequently, $m(O_2(\bar{H})) \leqslant 2$. This limits the possibilities for G. Either G has Lie rank 2 or $G \simeq \mathrm{Sp}(6, q)$, $O(7, q)'$, where in the latter cases $q \equiv 3 \pmod 4$. So $W \simeq D_6$, D_8, D_{12}, or the product of S_3 and a normal $Z_2 \times Z_2 \times Z_2$.

First suppose that G has Lie rank 2. If $W \simeq D_6$, then W has just one class of involutions, represented by s_1. Let $\bar{H}_1 = \bar{H} \cap C(\langle \bar{U}_{\pm\alpha_1} \rangle)$. Then $\bar{H}_1$ is cyclic of order $q - 1$ or $(q - 1)/3$.

So $C_{\bar{H}}(\bar{t})$ contains such a cyclic subgroup. However, $C_{\bar{H}}(\bar{t}) \leqslant (\bar{Y}_0 N_{\bar{Y}}(\bar{H}))^{\bar{g}}$ and $C_{\bar{H}}(\bar{t}) \cap (\bar{Y}_0\bar{H}_0)^{\bar{g}} = 1$ by (3.6). So $C_{\bar{H}}(\bar{t})$ is isomorphic to an Abelian subgroup of D_6, which contradicts the hypothesis that $q > 11$. Suppose $W \simeq D_8$ or D_{12}. Here W has three classes of involutions given by s_1, s_2, w_0, where w_0 is the longest word in s_1 and s_2. As before, if $i = 1, 2$, then $C_{\bar{H}}(\langle U_{\pm\alpha_i} \rangle)$ contains a cyclic group of order at least $(q - 1)/2$, or $q - 1$, according to $W \simeq D_8$ or D_{12}. This gives a contradiction as above. So suppose that $\bar{t}$ induces w_0 on $\bar{H}$. It is easy to see that w_0 centralizes $\Omega_1(O_2(\bar{H}))$, so by our choice of $\bar{T}$ we must have $O_2(\bar{H})$ cyclic. Therefore, $G \simeq \mathrm{Sp}(4, q)$ and $q \equiv 3 \pmod 4$.

Let $\langle \bar{j} \rangle = \Omega_1(O_2(\bar{H}))$, where $\langle j \rangle = Z(\langle U_{\pm\alpha} \rangle)$ for α a long root. Then we may assume that $j \in N(H^g)$ with j inverting $O(H^g)$ (symmetry). Let $H_1 = H \cap \langle U_{\pm\alpha} \rangle$ and let V be the natural module for G. So $V = V_1 \perp V_2$, where $[H_1, V_2] = 1$ and $\langle U_{\pm\alpha} \rangle$ induces $\mathrm{Sp}(2, q) = SL(2, q)$ on V_1. Also, j inverts H_1^g, so j acts on the 2-space, $[V, H_1^g] = V_1^g$. Then V_1^g intersects both V_1 and V_2 in a 1-space. The group $\langle H_1, H_1^g, j \rangle$ acts on the 3-space $V_0 = \langle V_1, V_1^g \rangle$. Now $V_0 = V_1 \perp \mathrm{rad}(V_0) = V_1 \perp (V_0 \cap V_2)$. So H_1^g stabilizes $V_0 \cap V_2$, as does j. But this is impossible since j interchanges the nontrivial eigenspaces of a generator of H_1^g.

Finally, we consider $G \simeq \mathrm{Sp}(6, q)$ or $O(7, q)'$, where $q \equiv 3 \pmod 4$. Then N is three copies of $D_{2(q-1)}$ extended by S_3. Now, $C_{O(\bar{H})}(\bar{t}) \leqslant (\bar{Y}_0 \bar{N}_{\bar{Y}}(\bar{H}))^{\bar{g}}$, while $C_{O(\bar{H})}(\bar{t}) \cap (\bar{Y}_0\bar{H})^{\bar{g}} = 1$. From the structure of W we have $|C_{O(H)}(t)| \leqslant 3$. In particular, t inverts $O_r(H)$ for all primes $r > 3$. Considering the action of N on the natural module for G we conclude that either t inverts H or $q - 1 = 2 \cdot 3^a$ and $|C_{\bar{H}}(\bar{t})| = 6$. The first case is out since $C_{O_2(\bar{H})}(\bar{t}) \simeq Z_2$, and the second is impossible since $q > 11$.

At this point we have $m(\bar{T}) > 1$, and by (3.7) we conclude that $q - 1 = 2^a$ for some integer $a \geqslant 4$ (as $q > 11$). We introduce the following notation: $A = \Omega_1(\bar{T})$, $B = \Omega_1(O_2(\bar{H})) \cap C(\bar{T})$, $X = \Omega_1(O_2(\bar{H}))$, and $Y_1 = N_Y(AB)$. We may assume that $B \leqslant N(X^g)$.

(3.9) *Assume* $B < X$.

(i) $A \simeq B$,

(ii) $Y_1 \geqslant \langle X, X^g \rangle$,

(iii) $[X, A] = B = [j, A]$, *for each* $j \in X - B$.

(iv) $C_X(a) = C_X(A) = B$, *for each* $a \in A^{\#}$.

Proof. For $a \in A^{\#}$, $C_X(a) \leqslant (\bar{Y}_0 N_Y(\bar{H}))^g$, so $[C_X(a), A] \leqslant X \cap (\bar{Y}_0\bar{H})^g = 1$ (by (3.6)). This proves (iv). On the other hand, $a \in A^{\#}$ implies that $[X, a] \leqslant C_X(a) = B$. Consequently, $[X, A] \leqslant B$ and so $X \leqslant N(AB) = N(A \times B)$. Similarly, $X^g = (N(AB))$, so (ii) holds.

If $x \in X - B$, then (iv) implies that $a \to [a, x]$ is a monomorphism from A into B. Thus, $|A| \leqslant |B|$ and since $M(\bar{T}) = m(\bar{A})$ is maximal, we have $|A| = |B|$, proving (i). In turn, this shows that $[A, x] = B$ for each $x \in X - B$. So (iii) holds.

(3.10) *Suppose* $B < X$, *let* $L = \langle X^{Y_1}\rangle AB$, *and let* $\tilde{L} = L/AB$. *Then* $\tilde{L}/O(\tilde{L}) \cong \tilde{X}$, $L_2(2^b)$, $U_3(2^b)$, $Sz(2^b)$, *each with* $b \geqslant 2$, $R(3^b)$, J_1, *or* $L_2(q_0)$ *for* $q_0 \equiv 3, 5 \pmod 8$.

Proof. The argument here is similar to the proof of (2.4) in [14]. We first show that $\tilde{X}$ is strongly closed in a Sylow 2-subgroup of $\tilde{L}$. For if not there exist $j_1, j_2 \in X^{\#}$, and $l \in L$ such that $[\tilde{j}_1^{\,l}, \tilde{j}_2] = 1$ and $\tilde{j}_1^{\,l} \notin \tilde{X}$. By (3.6) and (3.9)(i), $AB = B^l \times B$. But also, $j_1^{\,l} \in N(C_{AB}(j_2)) = B$. So $|C_{AB}(j_1^{\,l})| = |B^l|\,|C_B(j_2)| > |B^l| = |B| = |C_{AB}(j_1)|$, and this is impossible. Thus, the claim is proved and we can apply Goldschmidt's theorem [10]. We conclude that $\tilde{L}/O(\tilde{L})$ is a commuting product of $\tilde{X} \cap O_2(\tilde{L})$ and components isomorphic to $L_2(2^b)$, $U_3(2^b)$, $Sz(2^b)$, $R(3^b)$, J_1, and $L_2(q_0)$, $q \equiv 3, 5 \pmod 8$. Moreover, each of these factors is generated by elements in conjugates of $\tilde{X}$. If there is more than one factor, then we can choose $\tilde{j}_1^{\,g}$ and $\tilde{j}_2$, as before, and obtain a contradiction.

(3.11) $B = X$.

Proof. Suppose $B < X$. Since $O_2(\bar{H}) > X$, we choose $\bar{h} \in O_2(\bar{H}) \cap N(XA)$ with $\bar{h} \notin X$ and $\bar{h}^2 \in X$. Then $\bar{h}$ acts on $\mathrm{Inv}(XA) = X^{\#} \cup (AB)^{\#}$. Since $\bar{h} \in C(X)$ we conclude that $h \in N_Y(AB) = Y_1$. Then $[A, \bar{h}] \leqslant AB \cap O_2(\bar{H}) = B$, so $\bar{h}$ centralizes AB/B and B. It follows that $\bar{h}^2 \in C_X(AB) = B$ and $\bar{h}AB$ is an involution in Y_1/AB. Let L be as in (3.10).

We claim that $C_{AB}(\bar{h}) = B$. Otherwise, $\bar{h} \in C(a)$ for some $a \in A^{\#}$ and $\bar{h} \in N(\overline{Y_0H})^g)$. Hence $[\bar{h}, A] \leqslant \bar{H} \cap (\overline{Y_0H})^g = 1$. Setting $V = (\overline{Y_0X})^g/\bar{Y}_0^{\,g}$ this says that $C_V(\bar{h}) = C_V(\bar{h}^2)$, which is only possible if $\bar{h}^2 \in C(V)$. But this implies that $T = X^g$, so $B = X$, against our assumptions.

Write $D = O(\tilde{L}) = C_D(x)\, C_D(\bar{h})\, C_D(x\bar{h})$, where $x \in X - B$. It follows that $D \leqslant N(B)$. Indeed, $C_D(x)$ normalizes $C_{AB}(x) = B$, and similarly for $C_D(\bar{h})$ and $C_D(x\bar{h})$. As $D \trianglelefteq Y_1/AB$ we have $D \leqslant N(B^{\bar{y}})$, for each $y \in Y_1$.

Consider the action of $\bar{h}$ on $\tilde{L}$. By (3.10), $\tilde{X}$ is a maximal elementary

Abelian 2-group in $\tilde{L}$, so $\tilde{h}AB \notin \tilde{L}$. Since $[\tilde{h}, \tilde{X}] = 1$, either there exists $x \in X$ with $x\tilde{h} \in C(\tilde{L}/O(\tilde{L})$ or $\tilde{L}/O(\tilde{L}) \simeq U_3(2^a)$ and $\tilde{h}$ induces a field (graph) automorphism of order 2 (see (19.4) of [4]). In any case, there is an element $x \in X$ such that $x\tilde{h} \in C(\langle \tilde{X}, \tilde{X}^{\tilde{y}} \rangle)$ for some $y \in Y$, with $\tilde{X}^{\tilde{y}} \neq \tilde{X}$. Then $[X^{\tilde{y}}, x\tilde{h}]$ is contained in P, where $P/AB = O(\tilde{L})$. There exists $c \in P$ such that $\langle X^{\tilde{y}c}, x\tilde{h} \rangle$ is a 2-group. So $[X^{\tilde{y}c}, x\tilde{h}] \leqslant AB$ and $x\tilde{h}$ stabilizes $C_{AB}(X^{\tilde{y}c}) = B^{\tilde{y}c}$. By the above, $B^{\tilde{y}c} = B^{\tilde{y}}$. However, $B^{\tilde{y}} \cap B = 1$ and $C_{AB}(x\tilde{h}) = B$ (use the claim above, replacing $\tilde{h}$ by $x\tilde{h}$). This is a contradiction.

(3.12) $B < X$.

Proof. Suppose $B = X$. Let $S \in \mathrm{Syl}_2(\bar{N})$ with $O_2(\bar{H})A \leqslant S$. Let $Q \in \mathrm{Syl}_2(\bar{U}_{\pm\alpha})$ for a long root $\alpha \in \Sigma$, and let $\Delta = \{Q^g : Q^g \leqslant S\}$. Then any two distinct elements of Δ commute (see (2.5) of [3]), and we write $\langle \Delta \rangle = Q_1 \cdots Q_h \trianglelefteq S$. For some i, $Q_i \cap \bar{H}$ is the maximal cyclic subgroup of Q_i, so $Q_i \cap \bar{H}$ is cyclic of order $q - 1$. Also, $a \in A$ implies that $Q_i^a = Q_j$ for some j such that $Z(Q_i) = Z(Q_j)$.

Suppose $a \in A^{\#}$, $Q_i \cap \bar{H}$ is cyclic of order $q - 1$, and $[a, Q_i \cap \bar{H}] = 1$. Then $Q_i \cap \bar{H} \leqslant N_{\bar{Y}}(\bar{Y}_0\bar{H})^{\bar{g}}$ but $(Q_i \cap \bar{H}) \cap (\bar{Y}_0\bar{H})^{\bar{g}} = 1$. Also $[A, Q_i \cap \bar{H}] \leqslant \bar{H} \cap (\bar{Y}_0\bar{H})^{\bar{g}} = 1$. By symmetry, we conclude that $I/\bar{H} = C_{\bar{N}}(B)/\bar{H}$ has exponent divisible by $q - 1 = 2^a \geqslant 16$. However, if G has Lie rank less than 6, then 16 does not divide $\exp(W)$. So G must have Lie rank at least 6. Also, $I/\bar{H} \geqslant A\bar{H}/\bar{H} \simeq A$ and $I/H \trianglelefteq W$. From the structure of W we see that $W' \leqslant I/\bar{H}$, and this is impossible since the preimage of W' in $\bar{N}$ does not centralize $B = \Omega_1(O_2(\bar{H}))$.

Now, $Z(Q_i) \neq Z(Q_j)$ for $i \neq j$ unless $\bar{G}$ is an orthogonal group (recall, $PSp(4, q) = O(5, q)'$ and $PSU(4, q) = O^-(6, q)'$) in which case, for each i, there is a unique $j \neq i$ such that $Z(Q_i) = Z(Q_j)$. Choose i such that $\bar{H} \cap Q_i \simeq Z_{q-1}$. As $[A, Z(Q_i)] = 1$, either $A \leqslant N(\bar{H} \cap Q_i)$ or $\bar{G}$ is an orthogonal group and $N_A(\bar{H} \cap Q_i) = A_0$ with $| A : A_0 | \leqslant 2$. So if $\bar{G}$ is not an orthogonal group and $\bar{G}$ has Lie rank at least 3, then $m(A) \geqslant 3$ and $C_A(\bar{H} \cap Q_i) \neq 1$. This contradicts the previous paragraph. This also works for $\bar{G}$ an orthogonal group of Lie rank at least 4. Suppose $\bar{G}$ is an orthogonal group of Lie rank 3. Then we may assume that $A_0 \simeq Z_2 \times Z_2$ and A_0 acts faithfully on $\bar{H} \cap Q_i$. But then some element $a_0 \in A_0^{\#}$ centralizes a maximal subgroup of $\bar{H} \cap Q_i$. As in the previous paragraph this implies that W has exponent divisible by $(q - 1)/2 = 2^{a-1} \geqslant 8$, a contradiction.

Finally, we suppose that G has Lie rank 2. If $Q_i^c = Q_j \neq Q_i$ for some $c \in A^{\#}$, then $\bar{h}\bar{h}^c \in C_Y(c)$, where $\langle \bar{h} \rangle = \bar{H} \cap Q_i$. But this implies that W has exponent divisible by $| \bar{h}\bar{h}^c | = 2^{a-1} \geqslant 8$. This is impossible. So $A \leqslant N(\bar{H} \cap Q_i)$, and so some $c \in A^{\#}$ centralizes a maximal subgroup of $\bar{H} \cap Q_i$. This leads to the same contradiction. The proof of (3.12) is now complete.

Since (3.11) and (3.12) are contradictory, we have now proved the theorem.

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